

STAGNATION ZONES FOR \mathcal{A} -HARMONIC FUNCTIONS ON CANONICAL DOMAINS

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ABSTRACT. We study stagnation zones of \mathcal{A} -harmonic functions on canonical domains in the Euclidean n -dimensional space. Phragmén-Lindelöf type theorems are proved.

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1. INTRODUCTION

In this article we investigate solutions of the \mathcal{A} -Laplace equation on canonical domains in the n -dimensional Euclidean space.

Suppose that D is a domain in \mathbb{R}^n , and let $f: D \rightarrow \mathbb{R}$ be a function. For $s > 0$, a subset $\Delta \subset D$ is called s -zone (*stagnation zone with the deviation s*) of f , if there exists a constant C such that the difference between C and the function f is smaller than s on Δ . We may, for example, consider difference in the sense of the sup norm,

$$\|f(x) - C\|_{C(\Delta)} = \sup_{x \in \Delta} |f(x) - C| < s,$$

the L^p -norm

$$\|f(x) - C\|_{L^p(\Delta)} = \left(\int_{\Delta} |f(x) - C|^p d\mathcal{H}^n \right)^{1/p} < s,$$

or the Sobolev norm

$$\|f(x) - C\|_{W_p^1(\Delta)} = \left(\int_{\Delta} |\nabla f(x)|^p d\mathcal{H}^n \right)^{1/p} < s,$$

where \mathcal{H}^d is the d -dimensional Hausdorff measure in \mathbb{R}^n .

For discussion about the history of the question, recent results and applications, see [SS06, SS07].

Some estimates of stagnation zone sizes for solutions of the \mathcal{A} -Laplace equation on locally Lipschitz surfaces and behaviour of solutions in stagnation zones, were given in [Mik07]. In this paper we consider solutions of the \mathcal{A} -Laplace equation in subdomains of \mathbb{R}^n of a special form, canonical domains. In two-dimensional case, such domains are sectors and strips. In higher dimensions, they are conical and

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cylindrical regions. The special form of domains allows us to obtain more precise results.

Below we study stagnation zones of generalized solutions of the \mathcal{A} -Laplace equation

$$\operatorname{div} \mathcal{A}(x, \nabla f) = 0$$

(see [HKM93]) with boundary conditions of types (see definitions 1.6 and 1.10 below):

$$\langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G$$

and

$$f \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G$$

on canonical domains in the Euclidean n -dimensional space, where G is a closed subset of ∂D . We will prove Phragmén-Lindelöf type theorems for solutions of the \mathcal{A} -Laplace equation with such boundary conditions.

Canonical domains. Let $n \geq 2$. Fix an integer k , $1 \leq k \leq n$, and set

$$d_k(x) = \left(\sum_{i=1}^k x_i^2 \right)^{1/2}.$$

We call the set

$$B_k(t) = \{x \in \mathbb{R}^n : d_k(x) < t\}$$

a k -ball and

$$\Sigma_k(t) = \{x \in \mathbb{R}^n : d_k(x) = t\}$$

a k -sphere in \mathbb{R}^n . In particular, the symbol $\Sigma_k(0)$ denotes the k -sphere with the radius 0, i.e. the set

$$\Sigma_k(0) = \{x = (x_1, \dots, x_k, \dots, x_n) : x_1 = \dots = x_k = 0\}.$$

For every $0 < k < n$ we set

$$p_k(x) = \left(\sum_{j=k+1}^n x_j^2 \right)^{1/2}.$$

and

$$\Sigma_k^*(t) = \{x \in \mathbb{R}^n : p_k(x) = t\}, \quad t \geq 0.$$

Let $0 < \alpha < \beta < \infty$ be fixed, and let

$$D_{\alpha, \beta}^k = \{x \in \mathbb{R}^n : \alpha < p_k(x) < \beta\}.$$

For $k = n - 1$ we also assume that $x_n > 0$. Then for $k = n - 1$ the $D_{\alpha, \beta}^{n-1}$ is the a layer between two parallel hyperplanes, and for $1 \leq k < n - 1$ the boundary of the domain $D_{\alpha, \beta}^k$ consists of two coaxial cylindrical surfaces. The intersections $\Sigma_k(t) \cap D_{\alpha, \beta}^k$ are precompact for all $t > 0$. Thus, the functions $d_k(x)$ are *exhaustion functions* for $D_{\alpha, \beta}^k$.

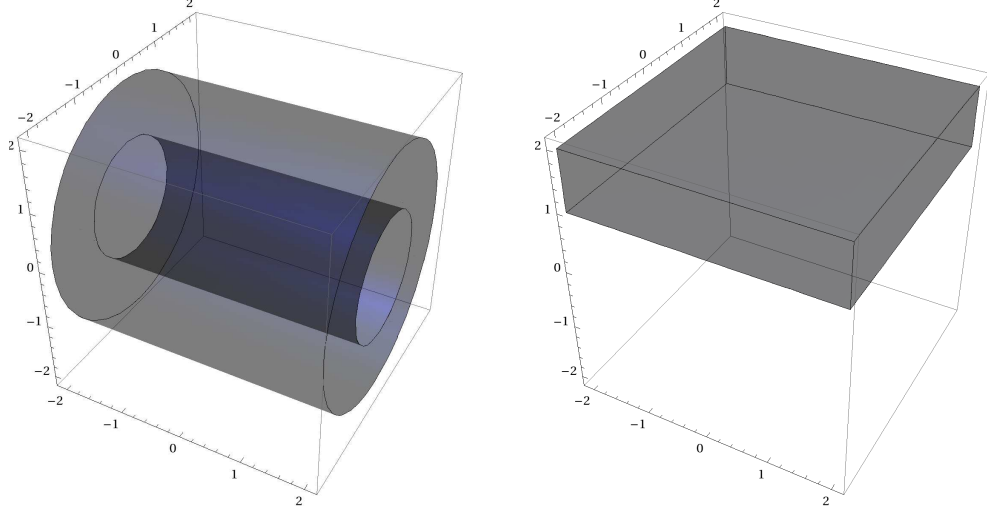


FIGURE: $D_{1,2}^1$ (left) and $D_{1,2}^2$ in \mathbb{R}^3 .

Structure conditions. Let D be a subdomain of \mathbb{R}^n and let

$$\mathcal{A}(x, \xi): \overline{D} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

be a vector function such that for a.e. $x \in \overline{D}$ the function

$$\mathcal{A}(x, \xi): \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is defined and continuous with respect to ξ . We assume that the function

$$x \mapsto \mathcal{A}(x, \xi)$$

is measurable in the Lebesgue sense for all $\xi \in \mathbb{R}^n$ and

$$(1.1) \quad \mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi), \quad \lambda \in \mathbb{R} \setminus \{0\}, \quad p \geq 1.$$

Suppose that for a.e. $x \in D$ and for all $\xi \in \mathbb{R}^n$ the following properties hold:

$$(1.2) \quad \nu_1 |\xi|^p \leq \langle \xi, \mathcal{A}(x, \xi) \rangle, \quad |\mathcal{A}(x, \xi)| \leq \nu_2 |\xi|^{p-1}$$

with $p \geq 1$ and some constants $\nu_1, \nu_2 > 0$. We consider the equation

$$(1.3) \quad \operatorname{div} \mathcal{A}(x, \nabla f) = 0.$$

An important special case of (1.3) is the Laplace equation

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} = 0.$$

As in [HKM93, Chapter 6], we call continuous weak solutions of the equation (1.3) \mathcal{A} -harmonic functions. However we should note that our definition of generalized solutions is slightly different from the definition given in [HKM93, p.56].

Frequencies. Fix $t \geq 0$ and $p \geq 1$. Let O be an open subset of $\Sigma_k^*(t)$ (with respect to the relative topology of $\Sigma_k^*(t)$), and let \mathcal{P} be a nonempty closed subset of ∂O . We set

$$(1.4) \quad \lambda_{p,\mathcal{P}}(O) = \inf_u \frac{\int_O |\nabla u|^p d\mathcal{H}^{n-1}}{\int_O u^p d\mathcal{H}^{n-1}},$$

where $u \in \text{Lip}_{\text{loc}}(O) \cap C^0(\overline{O})$ with $u|_{\mathcal{P}} = 0$. If $\mathcal{P} = \partial O$ we call $\lambda_p(O) \equiv \lambda_{p,\mathcal{P}}(O)$ the *first frequency* of the order $p \geq 1$ of the set O . If $\mathcal{P} \neq \partial O$ the quantity $\lambda_{p,\mathcal{P}}(O)$ is the *third frequency*.

The *second frequency* is the following quantity:

$$(1.5) \quad \mu_p(O) = \sup_C \inf_u \frac{\int_O |\nabla u|^p d\mathcal{H}^{n-1}}{\int_O (u - C)^p d\mathcal{H}^{n-1}},$$

where the supremum is taken over all constants C and $u \in \text{Lip}_{\text{loc}}(O) \cap C^0(\overline{O})$. See also Pólya and Szegő [PS51], Lax [Lax57].

Generalized boundary conditions. Suppose that D is a proper subdomain of \mathbb{R}^n . Let $\varphi: D \rightarrow \mathbb{R}$ be a locally Lipschitz function. We denote by $D_b(\varphi)$ the set of all points $x \in D$ at which φ does not have the differential. Let $U \subset D$ be a subset and let $\partial'U = \partial U \setminus \partial D$ be its boundary with respect to D . If $\partial'U$ is $(\mathcal{H}^{n-1}, n-1)$ -rectifiable, then it has locally finite perimeter in the sense of De Giorgi, and therefore a unit normal vector $\overline{\mathbf{n}}$ exists \mathcal{H}^{n-1} -almost everywhere on ∂U [Fed69, Sections 3.2.14, 3.2.15].

Let $D \subset \mathbb{R}^n$ be a domain and let $G \subset \partial D$ be a subset of the boundary of D . Define the concept of a generalized solution of (1.3) with zero boundary conditions on $\partial D \setminus G$. A subset $U \subset D$ is called *admissible*, if $\overline{U} \cap \overline{G} = \emptyset$ and U has a $(\mathcal{H}^{n-1}, n-1)$ -rectifiable boundary with respect to D .

Suppose that D is unbounded. Let $G \subset \partial D$ be a set closed in $\mathbb{R}^n \cup \{\infty\}$. We denote by (G, D) the collection of all subdomains $U \subset D$ with $\partial U \subset (D \cup (\partial D \setminus G))$ and $(\mathcal{H}^{n-1}, n-1)$ -rectifiable boundaries $\partial'U = \partial U \setminus \partial D$.

1.6. Definition. We say that a locally Lipschitz function $f: D \rightarrow \mathbb{R}$ is a generalized solution of (1.3) with the boundary condition

$$(1.7) \quad \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G,$$

if for every subdomain $U \in (G, D)$,

$$(1.8) \quad \mathcal{H}^{n-1}[\partial'U \cap D_b(f)] = 0,$$

and for every locally Lipschitz function $\varphi: \overline{U} \setminus G \rightarrow \mathbb{R}$ the following property holds:

$$(1.9) \quad \int_{\partial'U} \varphi \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_U \langle \mathcal{A}(x, \nabla f), \nabla \varphi \rangle d\mathcal{H}^n.$$

Here $\bar{\mathbf{n}}$ is the unit normal vector of $\partial'U$ and $d\mathcal{H}^n$ is the volume element on \mathbb{R}^n .

1.10. Definition. We say that a locally Lipschitz function $f: D \rightarrow \mathbb{R}$ is a generalized solution of (1.3) with the boundary condition

$$(1.11) \quad f \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G,$$

if for every subdomain $U \in (G, D)$ with (1.8), and for every locally Lipschitz function $\varphi: \overline{U} \setminus G \rightarrow \mathbb{R}$ the following property holds:

$$(1.12) \quad \int_{\partial'U} \varphi f \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_U \langle \mathcal{A}(x, \nabla f), \nabla(\varphi f) \rangle d\mathcal{H}^n.$$

In the case of a smooth boundary ∂D , and $f \in C^2(D)$, the relation (1.9) implies (1.3) with (1.7) everywhere on $\partial D \setminus G$. This requirement (1.12) implies (1.3) with (1.11) on $\partial D \setminus G$. See [Mik08, Section 9.2.1].

The surface integrals exist by (1.8). Indeed, this assumption guarantees that $\nabla f(x)$ exists \mathcal{H}^{n-1} -a.e. on $\partial'U$. The assumption that $U \in (G, D)$ implies existence of a normal vector $\bar{\mathbf{n}}$ for \mathcal{H}^{n-1} -a.e. points on $\partial'U$ [Fed69, Chapter 2 Section 3.2]. Thus, the scalar product $\langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle$ is defined and finite a.e. on $\partial'U$.

2. SAINT-VENANT PRINCIPLE

In this section, we will prove the Saint-Venant principle for solutions of the \mathcal{A} -Laplace equation. The Saint-Venant principle states that strains in a body produced by application of a force onto a small part of its surface are of negligible magnitude at distances that are large compared to the diameter of the part where the force is applied. This well known result in elasticity theory is often stated and used in a loose form. For mathematical investigation of the results of this type, see e.g. [BT08].

In this paper the inequalities of the form (2.3), (2.2) are called the Saint-Venant principle (see also, [OY77, BT08]). Here we consider only the case of canonical domains. We plan to consider the general case in another article.

Let $0 < k < n$. Fix a domain D_0 in \mathbb{R}^k with compact and smooth boundary, and write

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\}.$$

We write $\mathcal{P} = \{x \in \partial D : p_k(x) = \alpha\}$ and $\mathcal{Q} = \{x \in \partial D : p_k(x) = \beta\}$ and $G = \mathcal{P} \cup \mathcal{Q}$. Let $t, \tau \in (\alpha, \beta)$, $t < \tau$, and

$$\Delta^k(t, \tau) = \{x \in D : t < p_k(x) < \tau\}.$$

For $s \geq 0$ we set

$$\sigma^k(s) = \{x \in \Delta^k(0, \infty) : p_k(x) = s\}.$$

2.1. Theorem. *Let $\alpha < \tau' < \tau'' < \beta$, and let $0 < k < n$. If $f: D \rightarrow \mathbb{R}$ is a generalized solution of (1.3) with the generalized boundary condition (1.7) on $\partial D \setminus G$, then the inequality*

$$(2.2) \quad I_1(t, \tau') + C_1(t)/\nu_1 \leq (I_1(t, \tau'') + C_1(t)/\nu_1) \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^k(\tau)) d\tau \right].$$

holds for all $t \in (\alpha, \tau']$.

If $f: D \rightarrow \mathbb{R}$ is a generalized solution of (1.3) with the generalized boundary condition (1.11) then

$$(2.3) \quad I_1(t, \tau') + C_2(t)/\nu_1 \leq (I_1(t, \tau'') + C_2(t)/\nu_1) \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f(\tau)}^{1/p}(\sigma^k(\tau)) d\tau \right]$$

holds for all $t \in (\alpha, \tau']$. Here

$$I_1(t, \tau) = \int_{\Delta^k(t, \tau)} |\nabla f|^p d\mathcal{H}^n,$$

and

$$(2.4) \quad Z_f(\tau) = \{x \in \Sigma_k^*(\tau) \cap \partial D : \lim_{y \rightarrow x} f(y) = 0\}.$$

Proof. The case A. At first we consider the case in which f is a generalized solution of (1.3) with the generalized boundary condition (1.11) on $\partial D \setminus G$. It is easy to see that a.e. on $D_{\alpha, \beta}^k$,

$$|\nabla p_k(x)| = 1.$$

The domain $\Delta^k(t, \tau)$ belongs to (G, D) . Let $\varphi: \overline{U} \setminus G \rightarrow \mathbb{R}$ be a locally Lipschitz function. By (1.12) we have

$$\int_{\partial' \Delta^k(t, \tau)} \varphi f \langle \mathcal{A}(x, \nabla f), \mathbf{n} \rangle d\mathcal{H}^{n-1} = \int_{\Delta^k(t, \tau)} \langle \mathcal{A}(x, \nabla f), \nabla(\varphi f) \rangle d\mathcal{H}^n.$$

But

$$\partial' \Delta^k(t, \tau) = \sigma^k(t) \cup \sigma^k(\tau).$$

For $\varphi \equiv 1$, we have by (1.2) and (1.12)

$$\begin{aligned} \nu_1 I_1(t, \tau) &\leq \int_{\Delta^k(t, \tau)} \langle \mathcal{A}(x, \nabla f), \nabla f \rangle d\mathcal{H}^n \\ &= \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1} - \int_{\sigma^k(t)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1}, \end{aligned}$$

since $\bar{\mathbf{n}} = \nabla p_k(x)$ for $x \in \sigma^k(\tau)$ and $\bar{\mathbf{n}} = -\nabla p_k(x)$ for $x \in \sigma^k(t)$. We obtain

$$(2.5) \quad \nu_1 I_1(t, \tau) + C_2(t) \leq \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1}$$

where

$$C_2(t) = \int_{\sigma^k(t)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1}.$$

Note that we may also choose

$$(2.6) \quad \tilde{C}_2(\tau) = - \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1},$$

to obtain an inequality similar to (2.5).

Next we will estimate the right side of (2.5). By (1.2) and the Hölder inequality

$$\begin{aligned} & \left| \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1} \right| \\ & \leq \int_{\sigma^k(\tau)} |f| |\mathcal{A}(x, \nabla f)| d\mathcal{H}^{n-1} \leq \nu_2 \int_{\sigma^k(\tau)} |f| |\nabla f|^{p-1} d\mathcal{H}^{n-1} \\ & \leq \nu_2 \left(\int_{\sigma^k(\tau)} |f|^p d\mathcal{H}^{n-1} \right)^{1/p} \left(\int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1} \right)^{(p-1)/p}. \end{aligned}$$

By using (1.4) we may write

$$(2.7) \quad \int_{\sigma^k(\tau)} |f|^p d\mathcal{H}^{n-1} \leq \lambda_{p, Z_f(\tau)}^{-1}(\sigma^k(\tau)) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}$$

and

$$\left| \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1} \right| \leq \nu_2 \lambda_{p, Z_f(\tau)}^{-1/p}(\sigma^k(\tau)) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}.$$

By (2.5) and the Fubini theorem

$$\nu_1 I_1(t, \tau) + C_2(t) \leq \nu_2 \lambda_{p, Z_f(\tau)}^{-1/p}(\sigma^k(\tau)) \frac{dI_1}{d\tau}(t, \tau)$$

and

$$\frac{\nu_1}{\nu_2} \lambda_{p, Z_f(\tau)}^{1/p}(\sigma^k(\tau)) \leq \frac{dI_1}{d\tau}(t, \tau) \Big/ \left(I_1(t, \tau) + \frac{C_2(t)}{\nu_1} \right).$$

By integrating this differential inequality we have

$$\exp \left[\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f(\tau)}^{1/p}(\sigma^k(\tau)) d\tau \right] \leq \frac{I_1(t, \tau'') + C_2(t)/\nu_1}{I_1(t, \tau') + C_2(t)/\nu_1}$$

for arbitrary $\tau', \tau'' \in (\alpha, \beta)$ with $\tau' < \tau''$. We have shown that

$$(2.8) \quad I_1(t, \tau') + C_2(t)/\nu_1 \leq (I_1(t, \tau'') + C_2(t)/\nu_1) \exp \left[- \frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f(\tau)}^{1/p}(\sigma_p^k(\tau)) d\tau \right].$$

The case B. Now we assume that f is a generalized solution of (1.3) with the boundary condition (1.7) on $\partial D \setminus G$. Fix $t < \tau$. By choosing $\varphi \equiv 1$ in (1.9) we see that

$$\int_{\sigma^k(t) \cup \sigma^k(\tau)} \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = 0.$$

For an arbitrary constant C , we get from this and (1.9)

$$(2.9) \quad \int_{\sigma^k(t) \cup \sigma^k(\tau)} (f - C) \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_{\Delta^k(t, \tau)} \langle \mathcal{A}(x, \nabla f), \nabla f \rangle d\mathcal{H}^n.$$

Thus

$$\int_{\Delta^k(t, \tau)} \langle \mathcal{A}(x, \nabla f), \nabla f \rangle d\mathcal{H}^n \leq C_1(t) + \int_{\sigma^k(\tau)} |f - C| |\mathcal{A}(x, \nabla p_k(x))| d\mathcal{H}^{n-1},$$

where

$$C_1(t) = \int_{\sigma^k(t)} |f - C| |\mathcal{A}(x, \nabla p_k(x))| d\mathcal{H}^{n-1},$$

or

$$(2.10) \quad \nu_1 I_1(t, \tau) + C_1(t) \leq \nu_2 \int_{\sigma^k(\tau)} |f - C| |\nabla f|^{p-1} d\mathcal{H}^{n-1}.$$

As above, we obtain

$$(2.11) \quad \int_{\sigma^k(\tau)} |f - C| |\nabla f|^{p-1} d\mathcal{H}^{n-1} \leq \left(\int_{\sigma^k(\tau)} |f - C|^p d\mathcal{H}^{n-1} \right)^{1/p} \left(\int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1} \right)^{(p-1)/p}.$$

By using (1.5) we get

$$(2.12) \quad \left(\int_{\sigma^k(\tau)} |f - C_3|^p d\mathcal{H}^{n-1} \right)^{1/p} \leq \mu_p^{-1/p}(\sigma^k(\tau)) \left(\int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1} \right)^{1/p},$$

where $C_3 = C_3(f)$ is the constant from (1.5). Then by (2.11) and (2.12),

$$\int_{\sigma^k(\tau)} |f - C_3| |\nabla f|^{p-1} d\mathcal{H}^{n-1} \leq \mu_p^{-1}(\sigma^k(\tau)) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1},$$

and by (2.10) we have

$$\nu_1 I_1(t, \tau) + C_1(t) \leq \nu_2 \mu_p^{-1}(\sigma^k(\tau)) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}$$

or

$$\nu_1 I_1(t, \tau) + C_1(t) \leq \nu_2 \mu_p^{-1}(\sigma^k(\tau)) \frac{dI_1}{dt}(t, \tau).$$

By integrating this inequality we have shown that

$$(2.13) \quad I_1(t, \tau') + C_1(t)/\nu_1 \leq (I_1(t, \tau'') + C_1(t)/\nu_1) \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^k(\tau)) d\tau \right].$$

□

3. STAGNATION ZONES

Next we apply the Saint-Venant principle to obtain information about stagnation zones of generalized solutions of the equation (1.3). We first consider zones with respect to the Sobolev norm. Other results of this type follow immediately from well-known imbedding theorems.

Stagnation zones with respect to the W_p^1 -norm. We rewrite (2.2) and (2.3) in another form. Let $0 < k < n$ and let $0 < \alpha < \beta$. Fix a domain D_0 in \mathbb{R}^k with compact and smooth boundary, and write

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\}.$$

We write

$$p_k^*(x) = p_k(x) - \frac{\alpha + \beta}{2}.$$

For $x \in D_{\alpha, \beta}^k$ and

$$(3.1) \quad \beta^* = \frac{\beta - \alpha}{2}$$

we have

$$-\beta^* < p_k^*(x) < \beta^*,$$

and we denote

$$D_{\beta^*}^{*,k} = \{x \in \mathbf{R}^n : -\beta^* < p_k^*(x) < \beta^*\}.$$

Let $-\beta^* < \tau' \leq \tau'' < \beta^*$. We write

$$(3.2) \quad \Delta^{*,k}(\tau', \tau'') = \{x \in D : \tau' < p_k^*(x) < \tau''\}$$

and

$$I_2(\tau', \tau'') = \int_{\Delta^{*,k}(\tau', \tau'')} |\nabla f|^p d\mathcal{H}^n.$$

Let $0 < \tau' < \tau'' < \beta^*$. By (2.3) we have for $t \in (-\tau, \tau)$

$$I_2(t, \tau') + C_4(t)/\nu_1 \leq (I_2(t, \tau'') + C_4(t)/\nu_1) \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right],$$

where

$$(3.3) \quad Z_f^*(\tau) = \{x \in \partial D : p_k^*(x) = \tau \wedge \lim_{y \rightarrow x} f(y) = 0\}.$$

By choosing the estimate as in (2.6) we also have

$$I_2(-\tau', t) + \tilde{C}_4(t)/\nu_1 \leq (I_2(-\tau'', t) + \tilde{C}_4(t)/\nu_1) \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right],$$

where

$$(3.4) \quad \sigma^{*,k}(s) = \{x \in \Delta^{*,k}(-\infty, \infty) : p_k^*(x) = s\}.$$

By adding these inequalities and noting that $C_4(t) + \tilde{C}_4(t) = 0$ we obtain

$$\begin{aligned} I_2(-\tau', t) + I_2(t, \tau') &\leq (I_2(-\tau'', t) + I_2(t, \tau'')) \\ &\times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right] \right\}. \end{aligned}$$

Thus we have the estimate

$$(3.5) \quad \begin{aligned} I_2(-\tau', \tau') &\leq I_2(-\tau'', \tau'') \\ &\times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right] \right\}. \end{aligned}$$

Similarly, from (2.2) we obtain

$$(3.6) \quad \begin{aligned} I_2(-\tau', \tau') &\leq I_2(-\tau'', \tau'') \\ &\times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau)) d\tau \right] \right\}. \end{aligned}$$

From this we obtain the following theorem on stagnation W_p^1 -zones:

3.7. Theorem. *Let $0 < k < n$, $\beta > \alpha > 0$, and let $-\beta^* < \tau' \leq \tau'' < \beta^*$ where β^* is as in (3.1). If f is a solution of (1.3) on D with the generalized boundary condition (1.7) on $\partial D \setminus G$, where $G = \{x \in \partial D : p_k^*(x) = \pm\beta^*\}$ and*

$$\max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau)) d\tau \right] \right\} < s^{1/p},$$

or a solution of (1.3) on D with the generalized boundary condition (1.11) on $\partial D \setminus G$ and

$$\max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p,Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right] \right\} < s^{1/p},$$

then the subdomain $\Delta^{,k}(-\tau', \tau')$ is a s -zone with respect to the W_p^1 -norm i.e.,*

$$\int_{\Delta^{*,k}(-\tau', \tau')} |\nabla f|^p d\mathcal{H}^n < s,$$

where $\Delta^{,k}$ is as in (3.2).*

Stagnation zones with respect to the L^p -norm. Let $\beta > \alpha > 0$, and let $-\beta^* < \tau' \leq \tau'' < \beta^*$ where β^* is as in (3.1).

Denote by C_5 the best constant of the imbedding theorem from $W_p^1(D_{\beta^*}^{*,k})$ to $L^p(D_{\beta^*}^{*,k})$, i.e. in the inequality

$$\|g - C\|_{L^p(D_{\beta^*}^{*,k})} \leq C_5 \|g\|_{W_p^1(D_{\beta^*}^{*,k})},$$

if such constant exist (see Maz'ya [Maz85] or [AF03]). Then we obtain from (3.5),

$$(3.8) \quad \|f - C\|_{L^p(\Delta^{*,k}(-\tau', \tau'))}^p \leq C_5^p I_2(-\tau'', \tau'') \\ \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p,Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right] \right\},$$

and

$$(3.9) \quad \|f - C\|_{L^p(\Delta_{\tau'}^{*,k})}^p \leq C_5^p I_2(-\tau'', \tau'') \\ \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau)) d\tau \right] \right\}.$$

These relations can be used to obtain information about stagnation zones with respect to the L^p -norm. Namely, we have:

3.10. Theorem. *Let $0 < k < n$, and let*

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\},$$

where D_0 is a domain in \mathbb{R}^k with compact and smooth boundary. If f is a solution of (1.3) on D , with the generalized boundary condition (1.7) (or (1.11)) on $\partial D \setminus G$, where $G = \{x \in \partial D : p_k^(x) = \pm \beta^*\}$, and the right side of (3.8) (or (3.9)) is smaller than $s > 0$, then the domain $\Delta^{*,k}(-\tau', \tau')$ is a stagnation zone with the deviation s^p in the sense of the L^p -norm on D .*

Stagnation zones for bounded, uniformly continuous functions. Let $\beta > \alpha > 0$, and let $-\beta^* < \tau' \leq \tau'' < \beta^*$ where β^* is as in (3.1).

As before, denote by C_6 the best constant of the imbedding theorem from $W_p^1(D_{\beta^*}^{*,k})$ to $C(D_{\beta^*}^{*,k})$, i.e. in the inequality

$$(3.11) \quad \|g - C\|_{C(D_{\beta^*}^{*,k})} \leq C_6 \|g\|_{W_p^1(D_{\beta^*}^{*,k})},$$

if such constant exists. For example, if the domain $D_{\beta^*}^{*,k}$ is convex, then (3.11) holds for $p > n$ (see Maz'ya [Maz85] or [AF03, p.85]).

In this case from (3.5), (3.6) we obtain

$$(3.12) \quad \|f - C\|_{C(\Delta^{*,k}(-\tau', \tau'))} \leq C_6^p I_2(-\tau'', \tau'') \\ \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p, Z_f^*(\tau)}^{1/p}(\sigma^{*,k}(\tau)) d\tau \right] \right\},$$

and

$$(3.13) \quad \|f - C\|_{C(\Delta^{*,k}(-\tau', \tau'))} \leq C_6^p I_2(-\tau'', \tau'') \\ \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau)) d\tau \right] \right\}.$$

These relations can be used to obtain theorems about stagnation zones for bounded uniformly continuous functions.

3.14. Theorem. *Let $0 < k < n$. If f is a solution of (1.3), $p > n$, on D where D is as before with the generalized boundary condition (1.7) (or (1.11)) on $\partial D \setminus G$ where $G = \{x \in \partial D : p_k^*(x) = \pm \beta^*\}$, and the right side of (3.12) (or (3.13)) is smaller than $s > 0$, then the domain $\Delta^{*,k}(-\tau', \tau')$ is a stagnation zone with the deviation s in the sense of the norm $\|\cdot\|_{C^0(\Delta^{*,k}(-\tau', \tau'))}$.*

4. OTHER APPLICATIONS

Next we prove Phragmén-Lindelöf type theorems for the solutions of the \mathcal{A} -Laplace equation with boundary conditions (1.7) and (1.11).

Estimates for W_p^1 -norms. Let $\beta > \alpha > 0$, and let D_0 be a domain in \mathbb{R}^k with compact and smooth boundary. Write

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\}.$$

Suppose that β^* is as in (3.1). First we will prove some estimates of the W_p^1 -norm of a solution. Let f be a solution of (1.3) on $D_{\beta^*}^{*,k}$ with the generalized boundary condition (1.7) on $\partial D \setminus G$. Fix $0 < \tau' < \tau'' < \beta^*$ and estimate $\|f\|_{W_p^1(\Delta^{*,k}(-\tau', \tau'))}$.

Let $\psi : [\tau', \tau''] \rightarrow (0, \infty)$ be a Lipschitz function such that

$$(4.1) \quad \psi(\tau') = 1, \quad \psi(\tau'') = 0.$$

We choose

$$(4.2) \quad \phi(t) = \begin{cases} 1 & \text{for } |t| < \tau', \\ \psi(|t|) & \text{for } \tau' \leq |t| \leq \tau''. \end{cases}$$

The function $\varphi(x) = \phi(p_k^*(x))$ is admissible in Definition 1.6 for

$$U = \Delta^{*,k}(-\tau'', \tau'').$$

As in (2.9) we may by (1.9) write

$$\begin{aligned} \int_{\sigma^{*,k}(-\tau'') \cup \sigma^{*,k}(\tau'')} \phi^p(p_k^*(x))(f - C) \langle \mathcal{A}(x, \nabla f), \bar{\mathbf{n}} \rangle d\mathcal{H}^{n-1} \\ = \int_{\Delta^{*,k}(-\tau'', \tau'')} \langle \mathcal{A}(x, \nabla f), \nabla(\phi^p(p_k^*(x))(f - C)) \rangle d\mathcal{H}^n. \end{aligned}$$

By the construction of ϕ , (4.1) and (4.2), the surface integral is equal to zero, and we have

$$\begin{aligned} \int_{\Delta^{*,k}(-\tau'', \tau'')} \phi^p(p_k^*(x)) \langle \mathcal{A}(x, \nabla f), \nabla f \rangle d\mathcal{H}^n \\ = -p \int_{\Delta^{*,k}(-\tau'', \tau'')} \phi^{p-1}(p_k^*(x)) (f - C) \langle \mathcal{A}(x, \nabla f), \nabla \phi(p_k^*(x)) \rangle d\mathcal{H}^n. \end{aligned}$$

Thus by (1.2)

$$\begin{aligned} (4.3) \quad \nu_1 \int_{\Delta^{*,k}(-\tau'', \tau'')} \phi^p(p_k^*(x)) |\nabla f|^p d\mathcal{H}^n \\ \leq p\nu_2 \int_{\Delta^{*,k}(-\tau'', \tau'')} \phi^{p-1}(p_k^*(x)) |f - C| |\nabla f|^{p-1} |\nabla \phi(p_k^*(x))| d\mathcal{H}^n. \end{aligned}$$

Now we note that

$$|\nabla \phi(p_k^*(x))| = |\phi'(p_k^*(x))|$$

and by the Hölder inequality

$$\begin{aligned}
& \int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x)) |f - C| |\nabla f|^{p-1} |\nabla \phi(p_k^*(x))| d\mathcal{H}^n \\
&= \int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x)) |\nabla f|^{p-1} |\phi'(p_k^*(x))| |f - C| d\mathcal{H}^n \\
&\leq \left(\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x)) |\nabla f|^p d\mathcal{H}^n \right)^{(p-1)/p} \left(\int_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n \right)^{1/p}
\end{aligned}$$

From this inequality and (4.3) we obtain

$$\nu_1^p \int_{\Delta^{*,k}(-\tau',\tau')} \phi^p(p_k^*(x)) |\nabla f|^p d\mathcal{H}^n \leq p^p \nu_2^p \int_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n.$$

Because $\phi(p_k^*(x)) \equiv 1$ on $\Delta^{*,k}(-\tau',\tau')$ we have the following inequality:

$$(4.4) \quad \nu_1^p \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq p^p \nu_2^p \int_{\Delta^{*,k}(-\tau'',\tau'') \setminus \Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n.$$

Next we will find

$$\min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'') \setminus \Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n,$$

where the minimum is taken over all ψ in (4.2). We have

$$\begin{aligned}
& \int_{\Delta^{*,k}(-\tau'',\tau'') \setminus \Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n \\
&= \int_{-\tau''}^{-\tau'} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \\
&\quad + \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
 (4.5) \quad \min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'') \setminus \Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f(x) - C|^p d\mathcal{H}^n \\
 \leq \min_{\psi} \int_{-\tau''}^{-\tau'} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \\
 + \min_{\psi} \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \equiv A_1 + A_2.
 \end{aligned}$$

Because by the Hölder inequality

$$\begin{aligned}
 1 \leq \left(\int_{\tau'}^{\tau''} |\psi'(\tau)| d\tau \right)^p &\leq \left[\int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right] \\
 &\times \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{p-1},
 \end{aligned}$$

we have

$$\begin{aligned}
 \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\
 \leq \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1},
 \end{aligned}$$

and hence

$$A_2 \geq \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}.$$

It is easy to see that here the equality holds for a special choice of ψ . Thus

$$A_2 = \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}.$$

Similarly,

$$A_1 = \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}.$$

From (4.5) we obtain

$$\begin{aligned} \min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'') \setminus \Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f - C|^p d\mathcal{H}^n \\ \leq \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\ + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}. \end{aligned}$$

By using (4.4) we obtain the inequality:

$$\begin{aligned} p^{-p} \left(\frac{\nu_1}{\nu_2} \right)^p \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\ + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \end{aligned}$$

where C is an arbitrary constant. From this we obtain

$$(4.6) \quad \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq C_7 \max \left\{ \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\ \left. \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\}$$

where $C_7 = 2p^p(\nu_2/\nu_1)^p$.

Similarly, for the solutions of the \mathcal{A} -Laplace equation with the boundary condition (1.11) we may prove that

$$\begin{aligned} p^{-p} \left(\frac{\nu_1}{\nu_2} \right)^p \int_{\Delta^{*,k}(-\tau', \tau')} |\nabla f|^p d\mathcal{H}^n &\leq \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\ &\quad + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}. \end{aligned}$$

It follows that

$$(4.7) \quad \int_{\Delta^{*,k}(-\tau', \tau')} |\nabla f|^p d\mathcal{H}^n \leq C_7 \max \left\{ \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\ \left. \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\}.$$

Phragmén-Lindelöf type theorems I. We prove Phragmén-Lindelöf type theorems for cylindrical domains. Let $k = n - 1$. Fix a domain D_0 in \mathbb{R}^{n-1} with compact and smooth boundary. Consider the domain

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \dots, x_{n-1}) \in D_0\}.$$

Let $f_0: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.7) on ∂D .

Fix $\beta > \alpha > 0$, and let β^* be as in (3.1). Let $f(x) = f_0(x - \beta^* e_n)$, where e_n is the n :th unit coordinate vector, and let $0 < \tau' < \tau'' < \beta^* < \infty$. By (4.6)

$$\begin{aligned} \int_{\Delta^{*,k}(-\tau'', \tau'')} |\nabla f|^p d\mathcal{H}^n &\leq C_7 \max \left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\ &\quad \left. \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\}. \end{aligned}$$

By using (3.6) we obtain from this the inequality:

$$\begin{aligned}
I_2(-\tau', \tau') &\leq C_7 \max \left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\
&\quad \left. \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\} \\
&\quad \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,n-1}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p(\sigma^{*,n-1}(\tau)) d\tau \right] \right\}.
\end{aligned}$$

We observe that in this case

$$(4.8) \quad \mu_p(\sigma^{*,n-1}(\tau)) \equiv \mu_p(\sigma^{n-1}(0)),$$

and hence

$$\int_{\tau'}^{\tau''} \mu_p(\sigma^{*,n-1}(\tau)) d\tau = \mu_p(\sigma^{n-1}(0))(\tau'' - \tau').$$

It follows that

$$\begin{aligned}
(4.9) \quad I_2(-\tau', \tau') &\leq C_7 \max \left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\
&\quad \left. \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\} \\
&\quad \times \exp \left[-\frac{\nu_1}{\nu_2} \mu_p(\sigma^{n-1}(0))(\tau'' - \tau') \right].
\end{aligned}$$

By letting $\beta, \tau'' \rightarrow +\infty$ we obtain the following statement:

4.10. Theorem. *Fix a domain D_0 in \mathbb{R}^{n-1} with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \dots, x_{n-1}) \in D_0\},$$

and let $f: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.7) on ∂D . If for a constant C the right side of (4.9) goes to 0 as $\tau'' \rightarrow \infty$, then $f \equiv \text{const}$ on D .

Similarly for a solution f of (1.3) with (1.1) and (1.2), satisfying the boundary condition (1.11) we may write

$$(4.11) \quad I_2(-\tau', \tau') \leq C_7 \max \left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\ \left. \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\} \\ \times \max \left\{ \exp \left[-\frac{\nu_1}{\nu_2} \int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*}^{1/p}(\sigma^{*,n-1}(\tau)) d\tau \right], \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p,Z_f^*}^{1/p}(\sigma^{*,n-1}(\tau)) d\tau \right] \right\}.$$

However here we do not have any identity similar to (4.8). We have:

4.12. Theorem. *Fix a domain D_0 in \mathbb{R}^{n-1} with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \dots, x_{n-1}) \in D_0\},$$

and let $f: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.11) on ∂D . If the right side of (4.11) tends to 0 as $\tau'' \rightarrow \infty$, then $f \equiv 0$ on ∂D .

If $f(x) = 0$ everywhere on ∂D , then an identity similar to (4.8) holds in the following form:

$$(4.13) \quad \lambda_p^{1/p}(\sigma^{*,n-1}(\tau)) \equiv \lambda_p^{1/p}(\sigma^{n-1}(0)).$$

As above, we find

$$(4.14) \quad I_2(-\tau', \tau') \leq C_7 \max \left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p}, \right. \\ \left. \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \right\} \\ \times \exp \left[-\frac{\nu_1}{\nu_2} \lambda_p^{1/p}(\sigma^{n-1}(0))(\tau'' - \tau') \right].$$

Thus we obtain:

4.15. Corollary. *Fix a domain D_0 in \mathbb{R}^{n-1} with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \dots, x_{n-1}) \in D_0\},$$

and let $f: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition $f = 0$ on ∂D . If the right side of (4.14) tends to 0 as $\tau'' \rightarrow \infty$, then $f \equiv \text{const}$ on ∂D .

Phragmén-Lindelöf type theorems II. We prove Phragmén-Lindelöf type theorems for canonical domains of an arbitrary form. Let $1 \leq k < n - 1$. We consider a domain

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\},$$

where D_0 is a domain in \mathbb{R}^k with compact and smooth boundary. Let f be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.7) on ∂D .

Fix $\tau_0 > 0$. Let $\tau_0 < \tau' < \tau'' < \infty$. By (4.6) we may write

$$\int_{D_{0,\tau'}^k} |\nabla f|^p d\mathcal{H}^n \leq C_8 \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^k(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p},$$

where $C_8 = C_7/2$. As in (3.6) we obtain from (2.2) the estimate

$$\int_{D_{0,\tau_0}^k} |\nabla f|^p d\mathcal{H}^n \leq \int_{D_{0,\tau'}^k} |\nabla f|^p d\mathcal{H}^n \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau_0}^{\tau'} \mu_p(\sigma^k(\tau)) d\tau \right].$$

By combining these inequalities we obtain

$$(4.16) \quad \int_{D_{0,\tau_0}^k} |\nabla f|^p d\mathcal{H}^n \leq C_8 \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^k(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\ \times \exp \left[-\frac{\nu_1}{\nu_2} \int_{\tau_0}^{\tau'} \mu_p(\sigma^k(\tau)) d\tau \right].$$

The inequality (4.16) holds for arbitrary constant C and every $\tau'' > \tau'$. Thus the following statement holds:

4.17. Theorem. *Let $f: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.7) on ∂D , $1 \leq k < n - 1$. If for a constant C the right side of (4.16) tends to 0 as $\tau', \tau'' \rightarrow +\infty$, then $f \equiv \text{const}$ on D .*

If f satisfies (1.3) with (1.1), (1.2) and the boundary condition (1.11) on ∂D , then we have

$$(4.18) \quad \int_{D_{0,\tau_0}^k} |\nabla f|^p d\mathcal{H}^n \leq C_8 \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^k(\tau)} |f(x)|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{1-p} \\ \times \exp \left[- \frac{\nu_1}{\nu_2} \int_{\tau_0}^{\tau'} \lambda_{p,Z_f(\tau)}^{1/p}(\sigma^k(\tau)) d\tau \right].$$

We obtain:

4.19. Theorem. *Fix a domain D_0 in \mathbb{R}^k , where $1 \leq k < n - 1$, with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\},$$

and let $f: D \rightarrow \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary conditions (1.11) on ∂D . If for a constant C the right side of (4.18) tends to 0 as $\tau', \tau'' \rightarrow +\infty$, then $f \equiv 0$ on D .

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